

# Generalized P-Reducible $(\alpha, \beta)$ -Metrics with Vanishing S-curvature\*

A. Tayebi and H. Sadeghi

October 28, 2015

## Abstract

In this paper, we study one of the open problems in Finsler geometry which presented by Matsumoto-Shimada about the existence of P-reducible metric which is not C-reducible. For this aim, we study a class of Finsler metrics called generalized P-reducible metrics that contains the class of P-reducible metrics. We prove that every generalized P-reducible  $(\alpha, \beta)$ -metric with vanishing S-curvature reduces to a Berwald metric or C-reducible metric. It results that there is not any concrete P-reducible  $(\alpha, \beta)$ -metric with vanishing S-curvature.

**Keywords:** P-reducible metric, C-reducible metric, S-curvature.<sup>1</sup>

## 1 Introduction

In 1975, the well-known Physicist Y. Takano wrote a paper on physics which considered the field equation in a Finsler space and proposed certain geometrical problems in Finsler geometry [14]. He requested mathematicians to find some interesting special forms of hv-curvature from the standpoint of physics. In 1978, Matsumoto introduced the notion of P-reducible Finsler metrics as an answer to Takano which were a generalization of C-reducible Finsler metrics [7]. For a Finsler metric of dimension  $n \geq 3$ , he found some conditions under which the Finsler metric was P-reducible.

Since the study of hv-curvature becomes urgent necessity for the Finsler geometry as well as for theoretical physics, then Matsumoto-Shimada study the curvature properties of P-reducible metrics in [10]. They introduced the following open problem:

*Is there any concrete P-reducible metric which is not C-reducible?*

In [9], Matsumoto-Hōjō proves that  $F$  is C-reducible if and only if it is a Randers metric or Kropina metric. These metrics defined by  $F = \alpha + \beta$  and  $F = \alpha^2/\beta$ , respectively, where  $\alpha = \sqrt{a_{ij}y^i y^j}$  is a Riemann metric and  $\beta := b_i(x)y^i$  a 1-form on

---

\*Annales Polonici Mathematici, **114**(1) (2015), 67-79.

<sup>1</sup> 2010 Mathematics subject Classification: 53C60, 53C25.

a manifold  $M$ . The Randers metrics were introduced by G. Randers in the context of general relativity and have been widely applied in many areas of natural science, including biology, ecology, physics and psychology, etc [3]. The Kropina metric was introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal [1].

In [11], Numata introduced an interesting family of Finsler metrics called the Numata-type metrics. The Numata-type Finsler metrics are defined by  $F := \bar{F} + \eta$ , where  $\bar{F}(y) = \sqrt{g_{ij}(y)y^i y^j}$  is a locally Minkowskian metric and  $\eta = \eta_i(x)y^i$  a closed one-form on a manifold  $M$ .  $F$  is called a Randers change of  $\bar{F}$ . By a simple calculation, we get

$$C_{ijk} = \bar{C}_{ijk} + \frac{1}{2\bar{F}} \left\{ h_{ij} D_m + h_{jk} D_i + h_{ki} D_j \right\},$$

where  $D_i := \eta_i - \eta y_i / (\bar{F})^2$  and  $h_{ij} := F F_{ij}$  is the angular metric. Define  $\eta_{i|j}$  by  $\eta_{i|j} \gamma^j := d\eta_i - \eta_j \gamma_i^j$ , where  $\gamma^i := dx^i$  and  $\gamma_i^j := \Gamma_{ik}^j dx^k$  denote the linear connection form of  $\bar{F}$ . Put

$$\mathfrak{D}_{ij} := \frac{1}{2}(\eta_{i|j} + \eta_{j|i}).$$

Then the Landsberg curvature of  $F$  is given by

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij}, \quad (1)$$

where

$$\lambda = \frac{1}{2F} \mathfrak{D}_{ij} y^i y^j \quad \text{and} \quad a_i := \frac{1}{2F^2 \bar{F}^3} \left[ 2F \bar{F}^2 \mathfrak{D}_{ik} - 2F \mathfrak{D}_{kl} y^l y_i - (1 + \bar{F}^2) \mathfrak{D}_{kl} y^l D_j \right] y^k.$$

A Finsler metric  $F$  is called *generalized P-reducible* if its Landsberg curvature is given by (1), where  $a_i = a_i(x, y)$  and  $\lambda = \lambda(x, y)$  are scalar function on  $TM$ . Thus every Numata-type metric is a generalized P-reducible metric. By definition, if  $a_i = 0$  then  $F$  reduces to a general relatively isotropic Landsberg metric and if  $\lambda = 0$  then  $F$  is P-reducible. Thus the study of this class of Finsler spaces will enhance our understanding of the geometric meaning of P-reducible metrics.

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [12]. The Finsler metrics with vanishing S-curvature are some of important geometric structures which deserved to be studied deeply. Then it is a natural problem to study Finsler metrics with vanishing S-curvature [15].

An  $(\alpha, \beta)$ -metric is a Finsler metric of the form  $F := \alpha \phi(s)$ ,  $s = \beta/\alpha$ , where  $\phi = \phi(s)$  is a  $C^\infty$  on  $(-b_0, b_0)$ ,  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is a Riemannian metric and  $\beta = b_i(x)y^i$  is a 1-form on  $M$ . For example,  $\phi = c_1 \sqrt{1 + c_2 s^2} + c_3 s$  is called Randers type metric, where  $c_1 > 0$ ,  $c_2$  and  $c_3$  are constant. In this paper, we characterize generalized P-reducible  $(\alpha, \beta)$ -metrics with vanishing S-curvature and prove the following.

**Theorem 1.1.** *Let  $F = \alpha \phi(s)$ ,  $s = \beta/\alpha$ , be an  $(\alpha, \beta)$ -metric on a manifold  $M$ . Suppose that  $F$  is a generalized P-reducible metric with vanishing S-curvature. Then  $F$  is a Berwald metric or C-reducible metric.*

By Theorem 1.1, it follows that there is no concrete P-reducible  $(\alpha, \beta)$ -metric with vanishing S-curvature (see Lemma 3.5).

In this paper, we use the Berwald connection and the  $h$ - and  $v$ -covariant derivatives of a Finsler tensor field are denoted by “ $|$ ” and “ $,$ ” respectively.

## 2 Preliminaries

Let  $(M, F)$  be a Finsler manifold. Suppose that  $x \in M$  and  $F_x := F|_{T_x M}$ . We define  $\mathbf{C}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$  by  $\mathbf{C}_y(u, v, w) := C_{ijk}(y)u^i v^j w^k$  where

$$C_{ijk} := \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k},$$

and  $g_{ij} := \frac{1}{2} [F^2]_{y^i y^j}$ . The family  $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$  is called the Cartan torsion. It is well known that  $\mathbf{C} = 0$  if and only if  $F$  is Riemannian. For  $y \in T_x M_0$ , define mean Cartan torsion  $\mathbf{I}_y$  by  $\mathbf{I}_y(u) := I_i(y)u^i$ , where  $I_i := g^{jk} C_{ijk}$ .

For  $y \in T_x M_0$ , define the Matsumoto torsion  $\mathbf{M}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$  by  $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^i v^j w^k$  where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\},$$

and  $h_{ij} = g_{ij} - F_{y^i} F_{y^j}$  is the angular metric.  $F$  is said to be C-reducible if  $\mathbf{M}_y = 0$ .

**Lemma 2.1.** ([9]) A Finsler metric  $F$  on a manifold  $M$  of dimension  $n \geq 3$  is a Randers metric or Kropina metric if and only if  $\mathbf{M}_y = 0, \forall y \in TM_0$ .

A Finsler metric called semi-C-reducible if its Cartan tensor is given by following

$$C_{ijk} = \frac{p}{n+1} \{h_{ij} I_k + h_{jk} I_i + h_{ik} I_j\} + \frac{q}{\|\mathbf{I}\|^2} I_i I_j I_k, \quad (2)$$

where  $p = p(x, y)$  and  $q = q(x, y)$  are scalar function on  $TM$  satisfying  $p + q = 1$  and  $\|\mathbf{I}\|^2 = I^m I_m$  (see [8][16][17]).

**Lemma 2.2.** ([8]) Every non-Riemannian  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$  is semi-C-reducible.

The horizontal covariant derivatives of the Cartan torsion  $\mathbf{C}$  and mean Cartan torsion  $\mathbf{I}$  along geodesics give rise to the Landsberg curvature  $\mathbf{L}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$  and mean Landsberg curvature  $\mathbf{J}_y : T_x M \rightarrow \mathbb{R}$  which are defined by  $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k$  and  $\mathbf{J}_y(u) := J_i(y)u^i$ , respectively, where

$$L_{ijk} := C_{ijk|s} y^s, \quad J_i := I_{i|s} y^s.$$

The families  $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$  and  $\mathbf{J} := \{\mathbf{J}_y\}_{y \in TM_0}$  are called the Landsberg curvature and mean Landsberg curvature, respectively. A Finsler metric is called a Landsberg metric and weakly Landsberg metric if  $\mathbf{L} = 0$  and  $\mathbf{J} = 0$ , respectively.

A Finsler metric  $F$  on  $n$ -dimensional manifold  $M$  is called P-reducible if its Landsberg curvature is given by following

$$L_{ijk} = \frac{1}{n+1} \{ J_i h_{jk} + J_j h_{ik} + J_k h_{ij} \}.$$

It is easy to see that, every C-reducible metric is P-reducible. But the converse maybe is not true [6].

Given an  $n$ -dimensional Finsler manifold  $(M, F)$ , then a global vector field  $\mathbf{G}$  is induced by  $F$  on  $TM_0$ , which in a standard coordinate  $(x^i, y^i)$  for  $TM_0$  is given by  $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$ , where  $G^i = G^i(x, y)$  are called spray coefficients and given by following

$$G^i := \frac{1}{4} g^{il} \left[ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right], \quad y \in T_x M.$$

$\mathbf{G}$  is called the spray associated to  $F$ .

For  $y \in T_x M_0$ , define  $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$  by  $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i} |_x$ , where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$  is called the Berwald curvature and  $F$  is called a Berwald metric if  $\mathbf{B} = \mathbf{0}$ .

For an  $(\alpha, \beta)$ -metric, let us define  $b_{i|j}$  by  $b_{i|j} \theta^j := db_i - b_j \theta^j_i$ , where  $\theta^i := dx^i$  and  $\theta^j_i := \Gamma^j_{ik} dx^k$  denote the Levi-Civita connection form of  $\alpha$ . Let

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), & s_{ij} &:= \frac{1}{2}(b_{i|j} - b_{j|i}), \\ r_{i0} &:= r_{ij} y^j, & r_{00} &:= r_{ij} y^i y^j, & r_j &:= b^i r_{ij}, \\ s_{i0} &:= s_{ij} y^j, & s_j &:= b^i s_{ij}, & r_0 &:= r_j y^j, & s_0 &:= s_j y^j. \end{aligned}$$

Let  $G^i = G^i(x, y)$  and  $G^i_\alpha = G^i_\alpha(x, y)$  denote the coefficients of  $F$  and  $\alpha$  respectively in the same coordinate system. The following holds

$$G^i = G^i_\alpha + \alpha Q s^i_0 + (-2Q \alpha s_0 + r_{00}) \left( \Theta \frac{y^i}{\alpha} + \Psi b^i \right), \quad (3)$$

where

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, & \Delta &:= 1 + sQ + (b^2 - s^2)Q', \\ \Theta &:= \frac{Q - sQ'}{2\Delta}, & \Psi &:= \frac{Q'}{2\Delta}. \end{aligned}$$

The mean Landsberg curvature of an  $(\alpha, \beta)$ -metric  $F = \alpha\phi(s)$ , is given by following formula

$$\begin{aligned} J_i := & -\frac{1}{2\alpha^4\Delta} \left( \frac{2\alpha^2}{b^2-s^2} \left[ \frac{\Phi}{\Delta} + (n+1)(Q-sQ') \right] (r_0+s_0)h_i \right. \\ & + \frac{\alpha}{b^2-s^2} \left[ \Psi_1 + s\frac{\Phi}{\Delta} \right] (r_{00}-2\alpha Qs_0)h_i + \alpha \left[ -\alpha Q's_0h_i + \alpha Q(\alpha^2s_i - \bar{y}_is_0) \right. \\ & \left. \left. + \alpha^2\Delta s_{i0} + \alpha^2(r_{i0}-2\alpha Qs_0) - (r_{00}-2\alpha Qs_0)\bar{y}_i \right] \frac{\Phi}{\Delta} \right), \end{aligned} \quad (4)$$

where

$$\begin{aligned} \Psi_1 &:= \sqrt{b^2-s^2}\Delta^{\frac{1}{2}} \left[ \frac{\sqrt{b^2-s^2}}{\Delta^{\frac{3}{2}}} \right]', \\ h_i &:= \alpha b_i - s\bar{y}_i, \quad \bar{y}_i := a_{ij}y^j, \\ \Phi &:= -(Q-sQ')(n\Delta+1+sQ) - (b^2-s^2)(1+sQ)Q''. \end{aligned}$$

For more details, see [2]. Then we have

$$\bar{J} := b^i J_i = -\frac{1}{2\alpha^2\Delta} \left\{ \Psi_1(r_{00}-2\alpha Qs_0) + \alpha\Psi_2(r_0+s_0) \right\}, \quad (5)$$

where

$$\Psi_2 := 2(n+1)(Q-sQ') + 3\frac{\Phi}{\Delta}.$$

For a Finsler metric  $F$  on an  $n$ -dimensional manifold  $M$ , the Busemann-Hausdorff volume form  $dV_F = \sigma_F(x)dx^1 \cdots dx^n$  is defined by

$$\sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left\{ (y^i) \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i} \Big|_x\right) < 1 \right\}}.$$

Let  $G^i(x, y)$  denote the geodesic coefficients of  $F$  in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[ \ln \sigma_F(x) \right],$$

where  $\mathbf{y} = y^i \frac{\partial}{\partial x^i} \Big|_x \in T_x M$ . If  $F$  is a Berwald metric then  $\mathbf{S} = 0$ .

In [4], Cheng-Shen characterize  $(\alpha, \beta)$ -metrics with isotropic S-curvature.

**Lemma 2.3.** ([4]) Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an non-Riemannian  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$  and  $b := \|\beta_x\|_\alpha$ . Suppose that  $F$  is not a Finsler metric of Randers type. Then  $F$  is of isotropic S-curvature,  $\mathbf{S} = (n+1)cF$ , if and only if one of the following holds

(a)  $\beta$  satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \quad (6)$$

where  $\varepsilon = \varepsilon(x)$  is a scalar function, and  $\phi = \phi(s)$  satisfies

$$\Phi = -2(n+1)k \frac{\phi \Delta^2}{b^2 - s^2}, \quad (7)$$

where  $k$  is a constant. In this case,  $\mathbf{S} = (n+1)cF$  with  $c = k\varepsilon$ .

(b)  $\beta$  satisfies

$$r_{ij} = 0, \quad s_j = 0. \quad (8)$$

In this case,  $\mathbf{S} = 0$ .

### 3 Proof of Theorem 1.1

**Lemma 3.1.** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a non-Randers type  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$ . Suppose that  $F$  has vanishing  $S$ -curvature. Then the following hold*

$$y_i s_0^i = 0, \quad (9)$$

$$y_i s_{0|0}^i = 0, \quad (10)$$

$$y_i b^j s_{j|0}^i = \phi(\phi - s\phi') s_0^j s_{j0}, \quad (11)$$

where  $y_i := g_{ij}y^j$ .

*Proof.* The following holds

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_j + b_j \alpha_i) + \rho_2 \alpha_i \alpha_j, \quad (12)$$

where  $\alpha_i := \alpha^{-1} a_{ij} y^j$  and

$$\rho := \phi(\phi - s\phi'), \quad (13)$$

$$\rho_0 := \phi\phi'' + \phi'\phi', \quad (14)$$

$$\rho_1 := -[s(\phi\phi'' + \phi'\phi') - \phi\phi'], \quad (15)$$

$$\rho_2 := s[s(\phi\phi'' + \phi'\phi') - \phi\phi']. \quad (16)$$

Then

$$y_i := \rho \bar{y}_i + \rho_0 b_i \beta + \rho_1 (b_i \alpha + s \bar{y}_i) + \rho_2 \bar{y}_i, \quad (17)$$

where  $\bar{y}_i := a_{ij} y^j$ . Since  $\bar{y}_i s_0^i = 0$  then by (8) we get  $b_i s_0^i = 0$ . Thus (17) implies that

$$y_i s_0^i = 0. \quad (18)$$

Since  $y_{i|0} = 0$ , then by (18) it follows that

$$y_i s_{0|0}^i = 0. \quad (19)$$

By  $s_j = b^j s_j^i = 0$ , we have

$$0 = (b^j s_j^i)|_0 = b_{|0}^j s_j^i + b^j s_{j|0}^i = (r_0^j + s_0^j) s_j^i + b^j s_{j|0}^i \quad (20)$$

or equivalently

$$b^j s_{j|0}^i = -s_0^j s_j^i. \quad (21)$$

By (17) and (21), we get

$$y_i b^j s_{j|0}^i = -(\rho + \rho_1 s + \rho_2) s_0^j s_j^i = (\rho + \rho_1 s + \rho_2) s_0^j s_{j0}^i. \quad (22)$$

Since  $\rho_1 s + \rho_2 = 0$ , then

$$y_i b^j s_{j|0}^i = \rho s_0^j s_{j0}^i = \phi(\phi - s\phi') s_0^j s_{j0}^i. \quad (23)$$

This completes the proof.  $\square$

**Lemma 3.2.** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a non-Randers type  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$ . Suppose that  $F$  has vanishing S-curvature. Then the following hold*

$$b^j b^k b^l L_{jkl} = 0, \quad (24)$$

$$b^i J_i = 0. \quad (25)$$

*Proof.* Since  $F$  has vanishing S-curvature, then (3) reduces to following

$$G^i = G_\alpha^i + \alpha Q s_0^i. \quad (26)$$

Taking third order vertical derivations of (26) with respect to  $y^j$ ,  $y^l$  and  $y^k$  yields

$$\begin{aligned} B_{jkl}^i = & s_l^i \left[ Q \alpha_{jk} + Q_k \alpha_j + Q_j \alpha_k + \alpha Q_{jk} \right] \\ & + s_j^i \left[ Q \alpha_{lk} + Q_k \alpha_l + Q_l \alpha_k + \alpha Q_{lk} \right] \\ & + s_k^i \left[ Q \alpha_{jl} + Q_j \alpha_l + Q_l \alpha_j + \alpha Q_{jl} \right] \\ & + s_0^i \left[ \alpha_{jkl} Q + \alpha_{jk} Q_l + \alpha_{lk} Q_j + \alpha_{lj} Q_k \right. \\ & \left. + \alpha Q_{jkl} + \alpha_l Q_{jk} + \alpha_j Q_{lk} + \alpha_k Q_{jl} \right] \end{aligned} \quad (27)$$

Multiplying (27) with  $y_i$  and using (9) implies that

$$\begin{aligned} -2L_{jkl} = & y_i s_l^i \left[ Q \alpha_{jk} + Q_k \alpha_j + Q_j \alpha_k + \alpha Q_{jk} \right] \\ & + y_i s_j^i \left[ Q \alpha_{lk} + Q_k \alpha_l + Q_l \alpha_k + \alpha Q_{lk} \right] \\ & + y_i s_k^i \left[ Q \alpha_{jl} + Q_j \alpha_l + Q_l \alpha_j + \alpha Q_{jl} \right]. \end{aligned} \quad (28)$$

By (8), we have  $s_j = b^j s_{ij} = 0$ . Then, multiplying (28) with  $b^j b^k b^l$  yields (24). By (5) and (8), we get (25).  $\square$

**Lemma 3.3.** *Let  $(M, F)$  be a generalized P-reducible Finsler manifold. Then the Matsumoto torsion of  $F$  satisfy in following*

$$M_{ijk|s}y^s = \lambda(x, y)M_{ijk}. \quad (29)$$

*Proof.* Let  $F$  be a generalized P-reducible metric

$$L_{ijk} = \lambda C_{ijk} + a_i h_{jk} + a_j h_{ki} + a_k h_{ij}. \quad (30)$$

Contracting (30) with  $g^{ij} := (g_{ij})^{-1}$  and using the relations  $g^{ij}h_{ij} = n - 1$  and  $g^{ij}(a_i h_{jk}) = g^{ij}(a_j h_{ik}) = a_k$  implies that

$$J_k = \lambda I_k + (n + 1)a_k. \quad (31)$$

Then

$$a_i = \frac{1}{n + 1}J_i - \frac{\lambda}{n + 1}I_i. \quad (32)$$

Putting (32) in (30) yields

$$\begin{aligned} L_{ijk} &= \lambda C_{ijk} + \frac{1}{n + 1}\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} \\ &\quad - \frac{\lambda}{n + 1}\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\}. \end{aligned} \quad (33)$$

By simplifying (33), we get (29).  $\square$

**Lemma 3.4.** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a non-Randers type  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$ . Suppose that  $F$  is a generalized P-reducible metric with vanishing S-curvature. Then  $F$  is a P-reducible metric.*

*Proof.* Let  $F$  be a generalized P-reducible metric. By Lemma 3.3, we have

$$L_{ijk} - \frac{1}{n + 1}(J_i h_{jk} + J_j h_{ik} + J_k h_{ij}) = \lambda \left[ C_{ijk} - \frac{1}{n + 1}(I_i h_{jk} + I_j h_{ik} + I_k h_{ij}) \right]. \quad (34)$$

Contracting (34) with  $b^i b^j b^k$  and using (24) and (25), implies that

$$\lambda \left[ b^i b^j b^k C_{ijk} - \frac{3}{n + 1}(b^i I_i)(b^j b^k h_{jk}) \right] = 0. \quad (35)$$

By (35), we get two cases as follows:

**Case (1):**  $\lambda = 0$ . In this case,  $F$  reduces to a P-reducible metric.

**Case (2):**  $\lambda \neq 0$ . In this case, by (35) we get

$$b^i b^j b^k C_{ijk} = \frac{3}{n + 1}(b^i I_i)(b^j b^k h_{jk}). \quad (36)$$



Multiplying (2) with  $b^i b^j b^k$  gives

$$b^i b^j b^k C_{ijk} = \frac{3p}{n+1} (b^i I_i) (b^j b^k h_{jk}) + \frac{q}{\|\mathbf{I}\|^2} (b^i I_i)^3. \quad (37)$$

By (36) and (37), it follows that

$$\frac{3q}{n+1} (b^i I_i) \left[ b^j b^k h_{jk} - \frac{(n+1)(b^m I_m)^2}{3\|\mathbf{I}\|^2} \right] = 0. \quad (38)$$

By (38), we get three cases as follows:

**Case (2a):** Let  $b^i I_i = 0$ . By a direct computation, we can obtain a formula for the mean Cartan torsion of  $(\alpha, \beta)$ - metrics as follows

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2} (\alpha b_i - s y_i). \quad (39)$$

If  $b^i I_i = 0$ , then by contracting (39) with  $b^i$ , we get

$$\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^3} (b^2\alpha^2 - \beta^2) = 0. \quad (40)$$

By (40), we have  $\Phi = 0$  or  $\phi - s\phi' = 0$  which implies that  $\mathbf{I} = 0$  and then  $F$  is a Riemannian metric. This is a contradiction with our assumptions.

**Case (2b):** Suppose that the following holds

$$b^j b^k h_{jk} - \frac{n+1}{3\|\mathbf{I}\|^2} (b^i I_i)^2 = 0. \quad (41)$$

Since  $h_{jk} = g_{jk} - F^{-2} g_{jm} g_{kl} y^m y^l$ , then

$$b^j b^k h_{jk} = b^j b^k g_{jk} - \frac{1}{F^2} (g_{jk} b^j b^k)^2 \quad (42)$$

By (41) and (42), we obtain

$$b^j b^k \left[ g_{jk} - \frac{n+1}{3\|\mathbf{I}\|^2} I_j I_k \right] = \left[ \frac{1}{F} g_{jk} b^j b^k \right]^2. \quad (43)$$

Since  $y^i I_i = 0$ , then by (43), we get

$$\left[ \left( g_{ij} - \frac{(n+1)I_i I_j}{3\|\mathbf{I}\|^2} \right) b^i \frac{y^j}{F} \right]^2 = \left[ \left( g_{ij} - \frac{(n+1)I_i I_j}{3\|\mathbf{I}\|^2} \right) b^i b^j \right]^2. \quad (44)$$

Put

$$G_{ij} := g_{ij} - \frac{n+1}{3\|\mathbf{I}\|^2} I_i I_j.$$

It follows from (44) that

$$\left[G_{ij}b^i\frac{y^j}{F}\right]^2 = G_{ij}b^ib^j. \quad (45)$$

Since  $G_{ij}y^iy^j = F^2$ , then (45) implies that

$$\left[G_{ij}b^i\frac{y^j}{F}\right]^2 = \left[G_{ij}b^ib^j\right]\left[G_{ij}\frac{y^i}{F}\frac{y^j}{F}\right]. \quad (46)$$

By Cauchy-Schwartz type inequality and (46), we have

$$b^i = k\frac{y^i}{F}, \quad (47)$$

where  $k$  is a real constant. Multiplying (47) with  $b_i$  and  $\bar{y}_i$ , respectively, implies that

$$F = k\beta/b^2 \quad \text{and} \quad F = k\alpha^2/\beta. \quad (48)$$

By (48), it follows that  $(b^2 - s^2)\alpha^2 = 0$  which is a contradiction.

**Case (2c):** If  $q = 0$  then  $p = 1$  and from (2), it results that  $F$  is C-reducible. In any cases,  $F$  is a P-reducible Finsler metric.  $\square$

Now, we are going to consider P-reducible  $(\alpha, \beta)$ -metrics with vanishing S-curvature.

**Lemma 3.5.** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a non-Randers type  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$ . Suppose that  $F$  is a P-reducible metric with vanishing S-curvature. Then  $F$  reduces to a Berwald metric or C-reducible metric.*

*Proof.* The Landsberg curvature of an  $(\alpha, \beta)$ -metric is given by

$$L_{ijk} = \frac{-\rho}{6\alpha^5} \left\{ h_i h_j C_k + h_j h_k C_i + h_i h_k C_j + 3E_i T_{jk} + 3E_j T_{ik} + 3E_k T_{ij} \right\}, \quad (49)$$

where

$$h_i := \alpha b_i - s \bar{y}_i, \quad (50)$$

$$T_{ij} := \alpha^2 a_{ij} - \bar{y}_i \bar{y}_j, \quad (51)$$

$$C_i := (X_4 r_{00} + Y_4 \alpha s_0) h_i + 3 \Lambda D_i,$$

$$E_i := (X_6 r_{00} + Y_6 \alpha s_0) h_i + 3 \mu D_i,$$

$$D_i := \alpha^2 (s_{i0} + \Gamma r_{i0} + \Pi \alpha s_i) - (\Gamma r_{00} + \Pi \alpha s_0) \bar{y}_i$$

$$X_4 := \frac{1}{2\Delta^2} \left\{ -2\Delta Q''' + 3(Q - sQ')Q'' + 3(b^2 - s^2)(Q'')^2 \right\},$$

$$X_6 := \frac{1}{2\Delta^2} \left\{ (Q - sQ')^2 + [2(s + b^2 Q) - (b^2 - s^2)(Q - sQ')] Q'' \right\},$$

$$Y_4 := -2QX_4 + \frac{3Q'Q''}{\Delta},$$

$$Y_6 := -2QX_6 + \frac{(Q - sQ')Q'}{\Delta},$$

$$\Lambda := -Q'', \quad \mu := -\frac{1}{3}(Q - sQ'),$$

$$\Gamma := \frac{1}{\Delta}, \quad \Pi := \frac{-Q}{\Delta}.$$

For more details see [13]. Since  $r_{ij} = 0$  and  $s_i = 0$ , then (4) and (49) reduce to following

$$J_i = -\frac{\Phi}{2\alpha\Delta} s_{i0}, \quad (52)$$

$$L_{ijk} = V_{ij} s_{k0} + V_{jk} s_{i0} + V_{ki} s_{j0}, \quad (53)$$

where

$$V_{ij} := \frac{\rho}{2\alpha^3} \left[ Q'' h_i h_j + (Q - sQ') T_{ij} \right].$$

We shall divide the problem into two cases: (a)  $s_{i0} = 0$  and (b)  $s_{i0} \neq 0$ .

**Case (a):** Let  $s_{i0} = 0$ . In this case, by (52) and (53),  $F$  reduces to a Landsberg metric. By Shen' Theorem in [13],  $F$  reduces to a Berwald metric.

**Case (b):** Let  $s_{i0} \neq 0$ . Then by (52) and (53), we have

$$L_{ijk} = Z_{ij} J_k + Z_{jk} J_i + Z_{ki} J_j, \quad (54)$$

where  $Z_{ij} := -\frac{2\alpha\Delta}{\Phi} V_{ij}$ . Thus the Landsberg curvature of an  $(\alpha, \beta)$ -metric with vanishing S-curvature satisfies (54). Put

$$A := -\frac{\Delta\rho(Q - sQ')}{\Phi}, \quad B := -\frac{\Delta\rho Q''}{\Phi}.$$

Then by putting (50) and (51) in the formula of  $Z_{ij}$  it follows that

$$Z_{ij} = A a_{ij} + B b_i b_j - s B (b_i \alpha_j + b_j \alpha_i) - (A - s^2 B) \alpha_i \alpha_j. \quad (55)$$

By assumption,  $F$  is P-reducible

$$L_{ijk} = \frac{1}{n+1}(J_i h_{jk} + J_j h_{ik} + J_k h_{ij}), \quad (56)$$

where the angular metric  $h_{ij} := g_{ij} - F_{y^i} F_{y^j}$  is given by following

$$h_{ij} = \phi[\phi - s\phi']a_{ij} + \phi\phi'' b_i b_j - s\phi\phi''[b_i \alpha_j + b_j \alpha_i] - [\phi(\phi - s\phi') - s^2\phi\phi''] \alpha_i \alpha_j.$$

By (54) and (56), we obtain

$$(Z_{ij} - \frac{1}{n+1}h_{ij})J_k + (Z_{jk} - \frac{1}{n+1}h_{jk})J_i + (Z_{ik} - \frac{1}{n+1}h_{ik})J_j = 0. \quad (57)$$

Since  $\alpha_i s_0^i = 0$  and  $b_i s_0^i = 0$ , then we have

$$\begin{aligned} s_0^i s_0^j Z_{ij} &= -\frac{\Delta\rho}{\Phi}(Q - sQ')s_0^m s_{m0}, \\ s_0^i s_0^j h_{ij} &= \phi[\phi - s\phi']s_0^m s_{m0}, \\ s_0^i J_i &= -\frac{\Phi}{2\alpha\Delta}s_0^m s_{m0}. \end{aligned}$$

Therefore, contracting (57) with  $s_0^i s_0^j s_0^k$  implies that

$$\frac{1}{n+1}\phi[\phi - s\phi'] = A. \quad (58)$$

By (58), it follows that

$$Z_{ij} - \frac{1}{n+1}h_{ij} = \chi[b_i b_j - s(b_i \alpha_j + b_j \alpha_i) + s^2 \alpha_i \alpha_j], \quad (59)$$

where

$$\chi := B - \frac{1}{n+1}\phi\phi''.$$

Since  $J_i \neq 0$  and  $b^m J_m = 0$ , then by multiplying (57) with  $b^i b^j$ , we get

$$b^i b^j (Z_{ij} - \frac{1}{n+1}h_{ij}) = 0. \quad (60)$$

By contracting (59) with  $b^i b^j$  and considering (60), it follows that

$$\chi = 0. \quad (61)$$

(58) and (61) imply that

$$\frac{1}{n+1}\phi[\phi - s\phi'] = -\frac{\Delta\rho}{\Phi}(Q - sQ'), \quad (62)$$

$$\frac{1}{n+1}\phi\phi'' = -\frac{\Delta\rho}{\Phi}Q''. \quad (63)$$

By (62) and (63), we obtain

$$\phi - s\phi' = c(Q - sQ'), \quad (64)$$

where  $c$  is a non-zero real constant. Solving (64) implies that

$$Q = c_1\phi + c_2s, \quad (65)$$

where  $c_1 \neq 0$  and  $c_2$  are real constants. By (65), it follows that

$$c_2s^2 + 2c_1s\phi + 1 = d\phi^2, \quad (66)$$

where  $d$  is a real constant. We shall divide the problem into two cases: (b1)  $d \neq 0$  and (b2)  $d = 0$ .

**Subcase (b1):** If  $d \neq 0$ , then by (66) we have

$$\phi = \frac{c_1}{d}s + \sqrt{\left[\left(\frac{c_1}{d}\right)^2 + \frac{c_2}{d}\right]s^2 + 1} \quad (67)$$

which is a Randers-type metric. This is a contradiction.

**Subcase (b2):** If  $d = 0$ , then (66) yields

$$\phi = -\frac{1}{2c_1s} + \frac{c_2}{2c_1}s \quad (68)$$

which is a Randers change of a Kropina metric. It is known that, Kropina metrics are C-reducible. On the other hand, every Randers change of C-reducible metric is C-reducible [5]. Then the Finsler metric defined by (68) is C-reducible.  $\square$

**Proof of Theorem 1.1:** Every two-dimensional Finsler surface is C-reducible. For Finsler manifolds of dimension  $n \geq 3$ , by Lemmas 3.4 and 3.5, we get the proof.  $\square$

## References

- [1] L. Berwald, *On Cartan and Finsler geometries III, Two dimensional Finsler spaces with rectilinear extremals*, Ann of Math. **42**(1941), 84-112.
- [2] X. Cheng, *On  $(\alpha, \beta)$ -metrics of scalar flag curvature with constant S-curvature*, Acta. Math. Sinica, English Series, **26**(9) (2010), 1701-1708.
- [3] X. Cheng and Z. Shen, *Finsler Geometry, An Approach via Randers Spaces*, Springer-Verlag, 2012.
- [4] X. Cheng and Z. Shen, *A class of Finsler metrics with isotropic S-curvature*, Israel J. Math. **169**(2009), 317-340.

- [5] M. Matsumoto, *Projective Randers change of P-reducible Finsler spaces*, Tensor N. S. **59**(1998), 6-11.
- [6] M. Matsumoto, *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math. Kyoto Univ. **14**(1974), 477-498.
- [7] M. Matsumoto, *Finsler spaces with the hv-curvature tensor  $P_{hijk}$  of a special form*, Rep. Math. Phys. **14**(1978), 1-13.
- [8] M. Matsumoto, *Theory of Finsler spaces with  $(\alpha, \beta)$ -metric*, Rep. Math. Phys. **31**(1992), 43-84.
- [9] M. Matsumoto and S. Hōjō, *A conclusive theorem for C-reducible Finsler spaces*, Tensor. N. S. **32**(1978), 225-230.
- [10] M. Matsumoto and H. Shimada, *On Finsler spaces with the curvature tensors  $P_{hijk}$  and  $S_{hijk}$  satisfying special conditions*, Rep. Math. Phys. **12**(1977), 77-87.
- [11] S. Numata, *On the torsion tensors  $R_{jkh}$  and  $P_{hjk}$  of Finsler spaces with a metric  $ds = (g_{ij}(dx)dx^i dx^j)^2 + b_i(x)dx^i$* , Tensor (N.S.), **32**(1978), 27-32.
- [12] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances. Math. **128**(1997), 306-328.
- [13] Z. Shen, *On a class of Landsberg metrics in Finsler geometry*, Canad. J. Math. **61**(2009), 1357-1374.
- [14] Y. Takano, *On the theory of fields in Finsler spaces*, Intern. Symp. Relativity and Unified Field Theory, Calcutta, 1975.
- [15] A. Tayebi and B. Najafi, *On isotropic Berwald metrics*, Ann. Polon. Math. **103**(2012), 109-121.
- [16] A. Tayebi, E. Peyghan and B. Najafi, *On Semi-C-reducibility of  $(\alpha, \beta)$ -metrics*, Int. J. Geom. Meth. Modern. Phys, **9**(4) (2012), 1250038.
- [17] A. Tayebi and H. Sadeghi, *On Cartan torsion of Finsler metrics*, Publ. Math. Debrecen, **82**(2) (2013), 461-471.

Akbar Tayebi and Hassan Sadeghi  
 Department of Mathematics, Faculty of Science  
 University of Qom  
 Qom. Iran  
 Email: akbar.tayebi@gmail.com  
 Email: sadeghihassan64@gmail.com